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Inefficient rushes in auctions

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We analyze a setting common in privatizations, public tenders, and takeovers in which the ex post efficient allocation, i.e., the first best, is not implementable. Our first main result is that the open ascending auction is not second best because it is prone to *rushes*, i.e., all active bidders quitting simultaneously, that undermine its efficiency. Our second main result is that the second best can be implemented with a two-round auction used in real-life privatizations. We also show how this result generalizes using a survival auction with a novel tie-breaking rule.

Keywords. Privatization, efficiency, auctions, mechanism design, multi-round mechanisms.

JEL classification. D44, D82.

1. Introduction

The literature on auctions, going back to Vickrey's 1961 seminal paper, has remarked that the open ascending auction maximizes the expected gains from trade under fairly general conditions. This is the theoretical basis for the widespread support for the use of open ascending auctions in privatizations like the British 3G auction.[1] It is also a widely accepted argument in the analysis of markets that resemble an open ascending auction like takeovers.[2]

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[1] See Binmore and Klemperer (2002).

[2] Burkart and Panunzi (2008) write, "The bidding contest which yields the winning offer is commonly modeled as an English auction. In simple versions, bidders make offers and counter-offers at no cost,

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The above arguments are based on the observation that under mild conditions the open ascending auction implements the ex post efficient allocation, the first best, whenever it is implementable.[3,4] However, the first best is not implementable in many natural models of privatizations, and also public tenders and takeover contests if, as is often the case, there is an incumbent.[5] Suppose, for instance, franchise bidding for a service that was previously operated by one of the bidders, the incumbent. It seems plausible that the incumbent has private information about the demand of the service and lower setup costs but may have larger marginal costs than the other bidders, the entrants. In this case, the first best allocates to the incumbent when the demand is small and allocates to the entrant with largest value, otherwise, but this is not implementable because it is not a monotone allocation.

Our first main result, Proposition 2, shows that the open ascending auction is not second best in a general model with an incumbent that encompasses the above example. The reason is that the information endogenously disclosed along the auction prompts *rushes*, i.e., all active buyers drop their demand simultaneously, that under the usual uniformly random tie-breaking rule undermine the efficiency of the allocation; see Proposition 3. This result is consistent with some anecdotal evidence reported, for instance, by Binmore and Klemperer (2002) in the British 3G auction:

> The first withdrawal came in round 94 as the price of the cheapest license passed £2 billion ($3 billion), and four more withdrawals followed almost immediately.

Consider, as an illustration, our example above. The incumbent knows his value and thus bids until the price reaches it, as in a private value setting. Moreover, we show that the second best allocates the good to the entrant with the largest value in cases in which the incumbent has larger value than the entrants. Hence, implementing the second best requires that the entrants remain active at prices at which they make a loss when winning. This is what happens in equilibrium, and as a consequence, rushes arise after the incumbent quits and his type gets revealed. This explains the inefficiencies of our first main result as standard tie-breaking rules do not guarantee that the good

each offer incrementally higher than the previous, until the bidder with the highest valuation wins at a price equal to the valuation of the second highest bidder. Thus, competition leads to an efficient control allocation."

[3]An allocation is implementable if it is the equilibrium outcome of some mechanism. This definition differs from the notion of full implementation that requires the allocation to be the unique equilibrium outcome of some mechanism.

[4]Indeed, the cases in which the open ascending auction fails to be efficient when the first best is implementable are special. For instance, they do not arise in our model. More generally, Krishna (2003) shows that under the assumption that each bidder's value function is an increasing function that is additively separable into a private and a common component, which is a reasonable approximation, the open ascending auction implements the first best whenever the private value is increasing, which is a sufficient and (almost) necessary condition for the implementability of the first best. Choi et al. (2015) extend this result to a more general setting with insiders.

[5]See examples in Maskin (1992), Boone and Goeree (2009), and Hernando-Veciana and Michelucci (2011).

is allocated to the entrant with the largest value[6] as required by the second best. Our second main result, Proposition 5, is that the second best can be implemented with a two-round auction that corrects the inefficiencies of rushes while preserving the good properties of the open ascending auction. This second finding not only has a normative component, but also explains why our two-round auction is used in real life privatizations like ENI's (see Caffarelli 1998 and Perry et al. 2000), Telebras' privatization (see Dutra and Menezes 2002), and the 2007 Nigerian 800 MHz auction (see Nigerian-Communications-Commission 2007).

In our two-round auction, bidders submit sealed bids, the *initial bids*, in the first round and, in the second round, each of the two bidders submitting the largest initial bids, the *top bidders*, may raise their initial bids to determine their *final bids*. The winner is the top bidder, who submits the largest final bid and pays the other final bid. As in an open ascending auction, the second round is a second price auction with two bidders.[7] However, one key difference is that the initial bids of our two-round auction commit each entrant to a different minimum bid in the second round. Thus, entrants do not necessarily tie if they stick to their minimum prices, which explains our second main result.

The main difference between the open ascending auction and our two-round auction is that the bidders get continuous feedback about the prices at which rivals quit in the former auction but not in the latter. In this sense, our results show that disrupting the flow of information can be socially beneficial in certain environments.

Although the logic of our results is relatively transparent, our analysis has three challenges. First, one needs to establish that a rush leads to a tie. This is not obvious because the equilibrium conditions usually impose very little restrictions in information sets of the open ascending auctions in which there are more than two bidders active, as illustrated by Bikhchandani and Riley (1991). Hence, there could be an equilibrium in which only the entrant with the highest type remains active at the prices at which a rush could start. Second, one also needs to show that the random allocation typical of ties is inefficient. The challenge here is that Gershkov et al. (2013) and Hernando-Veciana and Michelucci (2014) provide auction examples in which the second best is a random allocation. We prove that this is not the case in our setting and provide a characterization of the second best that builds on the parallelism between the social problem and the problem solved by the entrants in auctions with a final round equivalent to a second price auction. Third, the equilibrium analysis of both the open ascending auction and our two-round auction does not follow from the usual analysis of open ascending auctions. This is because when the first best is not implementable, the equilibrium allocation is not ex post efficient, i.e., there is ex post regret, which requires a novel equilibrium characterization. Besides, this complicates the study of deviations in the early rounds of the auction.

[6]Nonstandard tie-breaking rules can solve the inefficiencies, as we show in the Supplementary Appendix, available in a supplementary file on the journal website, http://econtheory.org/supp/2513/supplement.pdf. This alternative is less appealing since it requires payments from the losers. Furthermore, the tie-breaking rule becomes an additional round of bidding, which makes it more similar to a two-round auction.

[7]The last round of the open ascending auction is not a second price auction, but it is strategically equivalent.

The simplicity of our two-round auction makes it well suited to be used in practice. However, two rounds may be insufficient to aggregate the information required to maximize efficiency in cases with more than one incumbent. Still, we can use the insights of our two-round auction to show that a survival auction with a novel tie-breaking rule outperforms the open ascending auction. The survival auction is a particular multi-round auction whose properties have been praised by Fujishima et al. (1999) and Kagel et al. (2007).

The most closely related paper is our earlier work, Hernando-Veciana and Michelucci (2011). It shows that the open ascending auction implements the second best if there are only two bidders. In this case, there are no ties after a rush and thus none of the conclusions derived here applies. Besides, our earlier work did not face the challenges we describe above because challenges one and three are a consequence of having more than two bidders, and challenge two does not arise when one restricts to deterministic allocations as in our earlier work.

Our results complement the analysis of Klemperer (1998), Perry et al. (2000), Klemperer (2002), Dutra and Menezes (2002), Levin and Ye (2008), Ye (2007), Boone and Goeree (2009), and Abraham et al. (2014). They provide settings in which a two-round auction gives greater expected revenue than open ascending auctions. Their arguments are unrelated to ours since, in their settings, two-round auctions do not do better than open ascending auctions in terms of social surplus maximization. For instance, Perry et al. (2000) consider a symmetric setting in which both the open ascending auction and their two-round auction implement the first best, whereas Boone and Goeree (2009) conjecture that a variation of the open ascending auction that uses proxy bids and a nonstandard tie-breaking rule could implement the second best.

The inefficiencies in our setting are due to rushes that arise when the first best is not implementable.[8] The efficiency of the open ascending auction when the first best is implementable has been studied by Maskin (1992), Krishna (2003), Dubra et al. (2009), Birulin and Izmalkov (2011), and Choi et al. (2015). The last paper is the closest as it also considers a setting with incumbents. The difference, though, is that their assumptions rule out the possibility that under the efficient allocation a change in an incumbent's type that increases his value moves the good from the incumbent to another bidder. This possibility is precisely the main focus of attention of our paper as our earlier example of franchise bidding illustrates.

Since we assume a one-dimensional type space, we do not explicitly consider the inefficiencies due to the multidimensionality of the type space; see Maskin (1992, 2000), Dasgupta and Maskin (2000), Eso and Maskin (2000), and Jehiel and Moldovanu (2001). In their case, the implementable allocations only depend on each bidder's type up to a particular one-dimensional projection, which is usually insufficient for the first best. Still, our analysis can be applied to the reduced form model in which each bidder's type

[8] Bulow and Klemperer (1994) have pointed out a similar effect to ours in open descending auctions: one bidder decreasing his demand may prompt other bidders to decrease their demands simultaneously,— leading to a tie. Their effect, however, requires multiunit sales and it is unrelated to the implementability of the first best.

is equal to these one-dimensional projections of the original model, as explained by Hernando-Veciana and Michelucci (2008).

Next, we define the theoretical setting, Section 2, and characterize the second best, Section 3. Section 4 provides a realistic setting in which our model applies. We analyze in Sections 5, 6, and 7 the open ascending auction, our two-round auction, and our survival auction, respectively. Section 8 concludes. The Appendix contains the main proofs and the Supplementary Appendix contains some additional results.

2. The model

One unit of an indivisible good is put up for sale to $n+1$ bidders, $n>1$, with quasilinear preferences in money. Bidder 1, the *incumbent* (male), puts monetary value $\hat{v}(s_1)\geq 0$ in getting the good and each bidder $i\neq 1$, the *entrants* (female), puts monetary value $v(s_i, s_1)\geq 0$, where s_1 and s_i denote, respectively, bidder 1's and bidder i's private information. We assume that s_j, $j\in\{1,2,\dots n+1\}$, is equal to the realization of an independent random variable, with distribution F_j, density f_j, and support normalized to be $[0,1]$. We assume that both $\hat{v}$ and v are increasing, strictly in the case of $\hat{v}$ and the first argument of v. To simplify the exposition, we assume that the functions v and $\hat{v}$ are continuous, although our results do not hinge on this assumption.

This model captures situations in which one bidder has inside information about a common component of the bidders' values. One example is the problem described in the Introduction, which is typical in privatizations and public tenders. Another example is the sale of a company where the common value component is the stand alone value of assets and the private values, bidders specific synergies. In this example, inside information arises when a firm has special links with the management team, for example, a white knight in the case of a hostile takeover; see Section C of our Supplementary Appendix.

Our assumptions are sufficiently general to model cases in which the ex post efficient allocation is not implementable while allowing for a tractable characterization of the second best. We discuss along the text the robustness of our results to our assumptions, in particular in footnote 12, in the third paragraph after Proposition 2 and in Section 7.

3. The first best and the second best

This section introduces the concepts of the first best and the second best, and provides a tractable characterization of the second best. These concepts and the characterization play a central role in the statement and discussion of our results. We start with some preliminary definitions.

An *allocation* is a measurable function p from the set of types $[0,1]^{n+1}$ into the $(n+1)$-dimensional simplex $\Delta(n+1)$ such that $p_i(s)$ denotes the probability of allocating the good to bidder i when the vector of types is s. We say that an allocation p is *first best* if it only allocates to the bidder with highest value. This is characterized by the following function.

Definition 1. Let $\rho:[0,1]\to[0,1]$ be implicitly defined by $v(\rho(s_1),s_1)=\hat{v}(s_1)$ if a solution exists. Otherwise, $\rho(s_1)\equiv 0$ if $v(0,s_1)>\hat{v}(s_1)$ and $\rho(s_1)\equiv 1$ if $v(1,s_1)<\hat{v}(s_1)$.

Our assumptions that v and $\hat{v}$ are continuous and that v is strictly increasing in its first argument imply that ρ is well defined. They also imply that that an entrant with type s_i has lower value than an incumbent with type s_1 if $s_i<\rho(s_1)$, whereas the opposite happens if $s_i>\rho(s_1)$. Consequently, an allocation p is first best if $p_1(s)=1$ when $\rho(s_1)>s_{(1)}\equiv\max\{s_j\}_{j\neq 1}$, and $p_i(s)=1$ for $i\neq 1$ when $s_i>\rho(s_1)$ and $s_i=s_{(1)}$.

We are interested in the set of *implementable allocations*. An allocation p is implementable if there exists a truthful equilibrium in a direct mechanism (p,x). A *direct mechanism* is a pair of measurable functions (p,x), where p is an allocation and $x:[0,1]^{n+1}\to\Delta(n+1)$ is a payment function. A direct mechanism (p,x) defines a game in which each bidder announces a type, and $p_i(s)$ denotes the probability that i gets the good and $x_i(s)$ gets her payment to the auctioneer when the vector of announced types is $s\in[0,1]^{n+1}$. A truthful equilibrium of a direct mechanism is a Bayesian Nash equilibrium in which all the bidders announce their true types. By the revelation principle, there is no loss of generality in restricting to the truthful equilibrium of direct mechanisms.

Lemma 1. *The first best is implementable if and only if ρ is weakly increasing.*

This condition is equivalent to the usual single crossing condition in auctions; see Krishna (2010, pp. 101 and 146).

We say that an allocation is *second best* if it maximizes the *expected social surplus*,

$$\int_{[0,1]^n}\int_{[0,1]}\Bigl(p_1(s)\hat{v}(s_1)+\sum_{i\neq 1}p_i(s)v(s_i,s_1)\Bigr)\,dF_1(s_1)\,dF_{-1}(s_{-1}),\tag{1}$$

subject to p implementable, where $F_{-1}(s_{-1})$ stands for $\prod_{j\neq 1}F_j(s_j)$. The next function is instrumental to characterize the second best.

Definition 2. Let $\phi:[0,1]\to[0,1]$ be

$$\phi(s_i)\in\operatorname*{arg\,max}_{q\in[0,1]}\int_0^q\bigl(v(s_i,s_1)-\hat{v}(s_1)\bigr)\,dF_1(s_1).\tag{2}$$

Lemma 2. *The function ϕ is increasing and uniquely defined by (2) a.e.*

To understand the intuitive meaning of the function ϕ, add the constant $\int_0^1\hat{v}(s_1)\,dF_1(s_1)$ to the objective function of (2), substitute q by $\tilde{\phi}(s_i)$, and take expectations with respect to s_i, assuming that s_i follows the distribution of the maximum of the entrants' types $s_{(1)}$ to get

$$\int_0^1\biggl(\int_0^{\tilde{\phi}(s_{(1)})}v(s_{(1)},s_1)\,dF_1(s_1)+\int_{\tilde{\phi}(s_{(1)})}^1\hat{v}(s_1)\,dF_1(s_1)\biggr)\,dF_{(1)}(s_{(1)}),$$

where $F_{(1)}(s_{(1)}) \equiv \prod_{i=2}^{n+1} F_i(s_{(1)})$. This expression is equal to the expected social surplus generated by an allocation that assigns the good to the entrant with the highest type $s_{(1)}$ if $\tilde{\phi}(s_{(1)}) \geq s_1$ and otherwise to the incumbent. We refer to this allocation as the *allocation associated to* $\tilde{\phi}$. Thus, the allocation associated to ϕ in (2) is the allocation that gives maximum expected social surplus among the allocations that are associated to the set of functions $\tilde{\phi} : [0, 1] \to [0, 1]$. We use this property next.

Proposition 1. *The second best allocation is unique (up to measure zero sets) and equal to the allocation associated to* ϕ.

In the proof of the proposition, we show that the second best allocation must be deterministic, be monotone, and only allocate to either the incumbent or the entrant with largest type;[9] thus, it is an allocation associated to an increasing function $\tilde{\phi} : [0, 1] \to [0, 1]$. Thus, the proposition follows from the property stated immediately above the proposition and the fact that monotone allocations are implementable.[10]

4. An economic application: Privatization with an incumbent

The purpose of this section is twofold. First, it provides a realistic setting in which the first best is not implementable.[11] Second, it presents an illustration of the different elements of our analysis.

The application here is a general version of the example discussed in the Introduction. A set of firms indexed by $i \in \{1, 2, \ldots, n+1\}$ compete for a privatized service that gives profits $\pi(A, C_i)$ after the firm incurs a setup cost K_i, and where A denotes a demand shifter and C_i denotes an individual variable cost shifter, e.g., firm i's marginal cost. Profit $\pi(A, C_i)$ is continuous, increasing in A, decreasing in C_i, and strictly submodular in (A, C_i): $\frac{\partial^2 \pi(A, C_i)}{\partial A \partial C_i} < 0$. The interpretation of submodularity is that the higher the demand, the more beneficial it is for the firm to have lower variable costs. This assumption is satisfied by most of the economic models that can give rise to the function π. An example is $\pi(A, C_i) = \frac{(A-C_i)^2}{2}$, which corresponds to the sale of a license to operate an unregulated monopoly with linear demand $Q(p) = A - p$ and constant marginal cost C_i.

Firm 1 is an incumbent already operating the service. It privately knows the demand for service A, has setup costs $K_1 = K$, and has a commonly known variable cost parameter $\overline{C}$. Each of the other firms—the entrants—has the same setup cost $K_i = K + \Delta$, which is greater than the incumbent's, i.e., $\Delta > 0$, but have a variable cost parameter $\overline{C} - \delta_i$, which is lower than the incumbent's, where $\delta_i \geq 0$ is firm i's private information.[12]

[9] An allocation p is deterministic if $p(s) \in \{0, 1\}^{n+1}$ and monotone if $p_i(s)$ is increasing in s_i for all i.

[10] Indeed, Section 6 provides a mechanism that implements the allocation associated to ϕ.

[11] The Supplementary Appendix has another application based on the model of privatizations of Boone and Goeree (2009). Note that although, their model differs from ours in that the incumbent has a two-dimensional signal, their Assumption 2 means that their analysis is formally equivalent to a model in which the incumbent has a one-dimensional signal equal to his value.

[12] The assumption that the entrants' marginal cost is private information but the incumbent's is commonly known is for simplicity. It makes the connection with our general model of Section 2 straightforward. Our analysis could be easily extended to a model in which both the incumbent's marginal cost and

In terms of our notation in the rest of the paper, the model of this section corresponds to $A = s_1$, where s_1 is a random variable with distribution F_1, and $\delta_i = s_i + \delta$ $(i \neq 1)$ for some $\delta > 0$ and s_i a random variable with distribution F_i. Moreover,

$$\hat{v}(s_1) = \pi(s_1, \overline{C}) - K,$$
$$v(s_i, s_1) = \pi(s_1, \overline{C} - \delta - s_i) - K - \Delta.$$

In this case, the equation that defines ρ is

$$0 = \pi(s_1, \overline{C} - \delta - s_i) - \pi(s_1, \overline{C}) - \Delta = -\int_{\overline{C}-\delta-s_i}^{\overline{C}} \left(\frac{\partial \pi(s_1, x)}{\partial x}\right) dx - \Delta. \tag{3}$$

Thus, in the case of interior solutions $\rho(s_1) \in (0, 1)$, the assumptions on the monotonicity and submodularity of π imply that ρ is a strictly decreasing function, which as a result of Lemma 1 implies that the first best allocation is not implementable. The second best is characterized by the solution to (2) whose maximand becomes

$$\int_0^q \big(\pi(s_1, \overline{C} - \delta - s_i) - \pi(s_1, \overline{C}) - \Delta\big)\, dF_1(s_1).$$

The submodularity of π implies that the integrand above is increasing in s_1 and, thus, the above expression is convex in q, which implies that the corresponding problem in (2) has a corner solution. This means that $\phi(s_i)$ is equal to either 0 or 1, depending on whether

$$\int_0^1 \big(\pi(s_1, \overline{C} - \delta - s_i) - \pi(s_1, \overline{C})\big)\, dF_1(s_1) - \Delta \tag{4}$$

is negative or positive, respectively. Since (4) is continuous and strictly increasing in s_i, our function ϕ is characterized by the value of s_i, say $\check{s}$, that makes (4) equal to 0. Thus

$$\phi(s_i) = \begin{cases} 0 & \text{if } s_i < \check{s}, \\ 1 & \text{if } s_i \geq \check{s}. \end{cases}$$

Figure 1 illustrates ρ and ϕ in the application of this section. The former function maps types of the incumbent in the vertical axis into a value for the highest type of the entrants in the horizontal axis: the first best allocates to the entrant with highest type above the graph of ρ and to the incumbent otherwise. The graph of the function ϕ has a similar interpretation with respect to the second best. The difference, though, is that ϕ maps values of the highest type of the entrants in the horizontal axis into incumbent's types in the vertical axis. Finally, note that $\check{s} \in [0, \rho(0))$ because $\pi(0, \overline{C} - \delta - \check{s}) - \pi(0, \overline{C}) - \Delta < 0$, since (4) is equal to zero at $s_i = \check{s}$ and its integrand is strictly increasing in s_1.

the demand shifter are the incumbent's private information. In this case, the incumbent's signal is multidimensional and we can apply the observation of Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Hernando-Veciana and Michelucci (2008) that in a multidimensional setting such as ours, the equilibrium allocation does not depend on more than a one-dimensional summary of each agent's private information. Thus, its equilibrium analysis could be done on a reduced form model in which each agent's type is equal to the one-dimensional summary of the original model. Our analysis could be applied to this reduced form model.

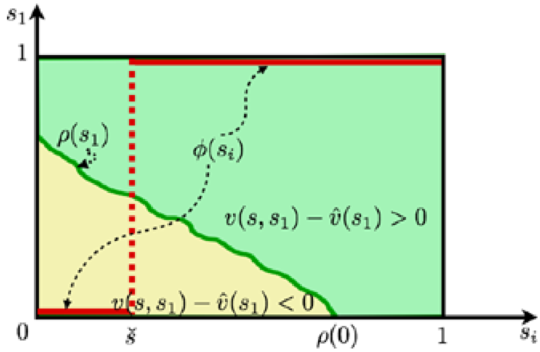

FIGURE 1. The functions ρ and ϕ in the economic application in Section 4.

5. The open ascending auction

In this section, we show that there is no equilibrium of the open ascending auction that implements the second best allocation under general conditions. Next, we show that the equilibrium yields rushes.

We assume the model of the open ascending auction described by Krishna (2003). This auction model is a variation of the Japanese auction proposed by Milgrom and Weber (1982) in which the identities of the active bidders are observable. Our auction is a dynamic game that starts with the price set equal to zero and all the bidders active. Both the price and the identities of the active bidders are publicly observable at any moment. The price increases continuously until one or more bidders decide to become inactive, i.e., quit. The decision to quit is irreversible. At that point, the price stops increasing and the following algorithm is implemented. If only one bidder remains active, this bidder gets the good at the current price. If all the bidders have quit, the good is allocated with equal probability among the bidders who last quit at the current price. Otherwise, the identify of the bidders who last quit is made public and bidders who have not quit yet can decide (independently and simultaneously) whether to quit. If some other bidders quit, we repeat the former algorithm. Otherwise, the price resumes increasing continuously until one or more other bidders quit. In this case, we repeat the former algorithm.

5.1 *The second best and the open ascending auction*

The next lemma gives some properties of the equilibrium bidding.

LEMMA 3. *In any undominated equilibrium of the open ascending auction, the following statements hold:*

(i) The incumbent with type s_1 quits at price $\hat{v}(s_1)$.

(ii) An entrant with type s_i finds it optimal to quit at price $\max\{v(s_i, s_1), \hat{v}(s_1)\}$ in information sets in which the incumbent has quit at price $\hat{v}(s_1)$.

(iii) An entrant with type s_i finds it optimal to quit at price $v(\phi(s_i))$ if all the other entrants have already quit when this price is reached.

Lemma 3(i) and (ii) follow from the property of open ascending auctions that a bidder who knows his value finds it optimal to remain active until the price reaches his value. The proof of (iii) uses that the entrant's optimal bid $b \in [\hat{v}(s_1), \hat{v}(1)]$ in the corresponding information sets maximizes

$$\int_{s_1}^{\hat{v}^{-1}(b)} \big(v(s_i, \tilde{s}_1) - \hat{v}(\tilde{s}_1)\big) \frac{dF_1(\tilde{s}_1)}{1 - F(s_1)},$$

which equals (2) if one changes $\hat{v}^{-1}(b)$ by q, multiplies by $1 - F(s_1)$, and adds $\int_0^{s_1} (v(s_i, \tilde{s}_1) - \hat{v}(\tilde{s}_1))\, dF_1(\tilde{s}_1)$.

Proposition 2. *There is no undominated equilibrium of the open ascending auction that implements the second best if there exists an open set of types for which $s_i < \rho(s_1)$ and $s_1 < \phi(s_i)$, i.e., that the first best allocates to the incumbent and the second best allocates to one of the entrants.*

The key elements of the proof of Proposition 2 are that the incumbent bids his value and that, for the vector of types in the proposition, the second best allocates to the entrant with the highest type whereas the first best allocates to the incumbent, and thus his value is larger than the entrants'. This means that implementing the second best requires that the entrant with the highest type wins at prices that are greater than her value, i.e., that there is ex post regret. In this case, all the active entrants realize that the price is greater than their values once the incumbent quits and, thus, they quit immediately after, i.e., there is a rush. If more than one entrant is active, there is a tie and the allocation is not second best because the tie-breaking rule allocates with equal probability. This argument does not necessarily mean that rushes arise in equilibrium as it is still possible that there are equilibria that do not implement the second best in which there are no rushes. However, Section 5.2 shows that rushes indeed arise in equilibrium.

The main challenge of the proof is to show that a rush leads to a tie. In principle, the huge multiplicity of equilibria typical of open ascending auctions (see Bikhchandani and Riley 1991) suggests that there could be an equilibrium in which all the entrants but the one with highest type quit before the price reaches the point at which a rush could start. Our proof shows that this is not possible. This is facilitated somehow by the assumption that the incumbent knows his value and thus has a unique weakly dominant strategy. However, we believe that our arguments could still be applied in two cases: when the elimination of weakly dominated strategies sufficiently narrows the incumbent's strategy or when one restricts to equilibria in which a bidder quits at a price equal to her expected value conditional on the information derived in equilibrium from the event that all the remaining bidders also quit at the same price. The latter restriction is consistent with most of the theoretical analyses of open ascending auctions, e.g., Milgrom and Weber (1982).

The conditions of Proposition 2 are met in our application in Section 4 by any function π if Δ, the difference in setup costs of the entrants and the incumbent, is neither so low that the first best always allocate to one of the entrants nor so high that the second best always allocate to the incumbent.[13]

5.2 *Equilibrium rushes*

In this subsection, we characterize the equilibrium of the open ascending auction under the conditions of Proposition 2 to illustrate that rushes, i.e., all active bidders quitting simultaneously, occur in equilibrium. To simplify the description of the equilibrium, we assume in this subsection that (I) all the entrants' types follow the same distribution F with density f, i.e., $F_j = F$ for all $j \neq 1$; (II) there are only two entrants, i.e., $n = 2$; (III) $\rho(0) > 0$; (IV) ρ is strictly decreasing, and (V) $\phi(s_i) = 1$ for all $s_i \in [0, 1]$.[14] We explain at the end of the section how our equilibrium characterization must be modified once these simplifying assumptions are relaxed. To guarantee the existence and uniqueness of γ below, we also assume that f and f_1 are bounded away from zero, and that v, $\hat{v}$, and f_1 are differentiable.

Our proposed equilibrium is defined by the bids in Lemma 3(i)–(iii); in the remaining information sets, an entrant with type s_i quits at price $\hat{v}(\gamma(s_i))$, where γ is the following increasing function.

Definition 3. Let γ be defined in $s_i \in [0, \bar{s}]$ as the unique continuous solution to[15]

$$\begin{aligned} &\beta \int_{\gamma(s_i)}^{1} \big(v(s_i, s_1) - \hat{v}(s_1)\big) \frac{dF_1(s_1)}{1 - F_1\big(\gamma(s_i)\big)} \\ &\quad + (1-\beta) \frac{F\big(\rho\big(\gamma(s_i)\big)\big) - F(s_i)}{1 - F(s_i)} \frac{v\big(s_i, \gamma(s_i)\big) - \hat{v}\big(\gamma(s_i)\big)}{2} = 0, \end{aligned} \tag{5}$$

starting at $\gamma(0) = 0$, where $\bar{s}$ is the point where the solution crosses the graph of ρ and

$$\beta \equiv \frac{\dfrac{f(s_i)}{\gamma'(s_i)}\big(1 - F_1\big(\gamma(s_i)\big)\big)}{\dfrac{f(s_i)}{\gamma'(s_i)}\big(1 - F_1\big(\gamma(s_i)\big)\big) + \big(1 - F(s_i)\big) f_1\big(\gamma(s_i)\big)}, \tag{6}$$

and let $\gamma(s_i) \equiv 1$ in $s_i \in (\bar{s}, 1]$.

The left panel of Figure 2 plots the function γ in a version of Figure 1 that satisfies assumptions (III)–(V): γ starts at $(0, 0)$ and satisfies (5) whenever $v(s_1, s_i) - \hat{v}(s_1) < 0$ and jumps to 1 afterward.

[13] The details are available in the Supplementary Appendix.

[14] We have formulated (III), (IV), and (V) in terms of the endogenous objects ρ and ϕ to make them more transparent. Since $v(s_i, 0)$ increases with s_i, Definition 1 implies that (III) is equivalent to $v(0, 0) - \hat{v}(0) < 0$. By Definition 1, (IV) is equivalent to $v(s_i, s_1) - \hat{v}(s_1)$ crossing zero at most once and from below as s_1 increases from 0 to 1. Since $v(s_i, s_1)$ is increasing in s_i, Definition 2 means that (V) is equivalent to $\int_0^1 (v(0, s_1) - \hat{v}(s_1))\, dF_1(s_1) > \int_0^q (v(0, s_1) - \hat{v}(s_1))\, dF_1(s_1)$, i.e., $\int_q^1 (v(0, s_1) - \hat{v}(s_1))\, dF_1(s_1) > 0$, for any $q \in [0, 1)$.

[15] The existence and uniqueness of γ follows from standard results in differential equations; see the Supplementary Appendix.

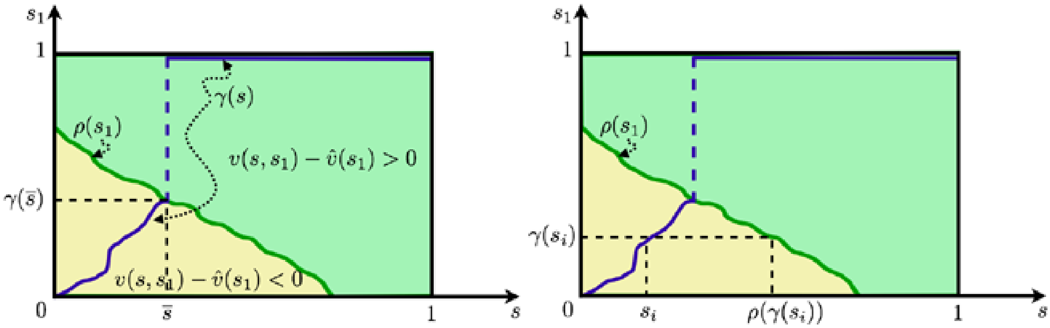

FIGURE 2. Illustration of the function γ and its role in the equilibrium.

The left hand side of (5) is equal to the expected utility of an entrant with type s_i who marginally outbids the bidder quitting first under the assumption that the bidders use our proposed bids. To see why, note that the lowest bid of the other bidders must be equal to $\hat{v}(\gamma(s_i))$, and the continuation play, and thus payoffs, depends on whether this lowest bid is the other entrant's bid or the incumbent's. These two cases have conditional probabilities β and $1-\beta$, respectively, where β is defined in (6).

If it is the other entrant who bids $\hat{v}(\gamma(s_i))$, the incumbent has a type higher than $\gamma(s_i)$, which is good news. In this case, our proposed bid function and assumption (V) imply that our entrant remains active until the incumbent quits, which gives an expected payoff equal to the expression multiplied by β in (5).

If it is the incumbent who bids $\hat{v}(\gamma(s_i))$, his type is $\gamma(s_i)$, which is bad news: since $(s_i, \gamma(s_i))$ lies in the region where $v(s_i, s_1) - \hat{v}(s_1) < 0$, our entrant's value $v(s_i, \gamma(s_i))$ is less than the current price $\hat{v}(\gamma(s_i))$. Thus, our entrant quits immediately after the incumbent. A similar argument also implies that the other entrant also quits immediately after the incumbent if her type lies in $(s_i, \rho(\gamma(s_i)))$; see the right panel of Figure 2. In this case, both entrants tie. Otherwise, our entrant is outbid. The first quotient after $(1-\beta)$ in (5) indicates the conditional probability of a tie; the second quotient indicates the expected losses in a tie.

In the region $s_i \in (\bar{s}, 1]$, our proposed strategy specifies that the entrant waits until either the incumbent or the other entrant quits. As happened in the previous paragraph, it is possible that our entrant incurs a loss when it is the incumbent who quits first. This occurs when the entrant's and incumbent's types lie below the graph of ρ. However, this loss is compensated by the gains when either the incumbent quits at a higher price or when it is the other entrant who quits first.

PROPOSITION 3. *Under assumptions (I)–(V), the following profile of strategies is an equilibrium:*

- *The incumbent quits at price $\hat{v}(s_1)$ when his type is s_1.*
- *An entrant with type s_i quits at the following prices:*
  - *Price $\hat{v}(\gamma(s_i))$ in information sets in which no bidder has quit yet, where γ is defined in (5).*

– *Price* $\max\{v(s_i, s_1), \hat{v}(s_1)\}$ *in information sets in which the incumbent quits at a price* $\hat{v}(s_1)$.

– *Price* $\hat{v}(\phi(s_i)) = \hat{v}(1)$ *in information sets in which the incumbent is the only other active bidder.*

In this equilibrium, there are rushes, i.e., all active bidders quit simultaneously. They occur when the vector of types (s_1, s_2, s_3) satisfies $s_1 < \min\{\gamma(s_2), \gamma(s_3)\}$ and $\max\{s_2, s_3\} < \rho(s_1)$. Since the entrants bid $\hat{v}(\gamma(s_i))$ and the incumbent bids $\hat{v}(s_1)$, the first condition means that the incumbent quits before any of the two entrants. Since $v(\rho(s_1), s_1) = \hat{v}(s_1)$ and v is an increasing function, the second condition means that both entrants quit immediately after the incumbent (by Lemma 3(ii)). To the extent that it is possible to construct examples in which both $\rho(s_1)$ and $\gamma(s_i)$ are arbitrarily close to 1 in $[0, 1)$, as we show in Proposition B of the Supplementary Appendix, the probability that there is a rush may be arbitrarily close to 1.

We now discuss the consequences of relaxing (I)–(V). Relaxing (I) implies a different equilibrium strategy for each entrant. This implies that several versions of the differential equation provided in (5), one per entrant, need to be considered. Relaxing (II) requires characterizing the bid behavior in information sets in which the incumbent and three or more entrants are active. The only complication is the notational burden. Assumptions (III)–(V) guarantee that the set of types for which the conditions of Proposition 2 apply lies at the bottom of the support of all the bidders' types, as in Figure 2. If there are several disconnected sets where the conditions of Proposition 2 apply, our characterization requires considering a version of the differential equations in (5) for each of these sets.

6. A two-round auction

In this section, we study a two-round auction that implements the second best. In the first round, all bidders submit an initial bid; in the second round, the bidders submitting the two highest initial bids—the top bidders—are allowed to revise their initial bids upward to make their final bids after their identity has been revealed publicly. The good is allocated to the top bidder who submits the highest final bid at a price equal to the final bid of the other top bidder.[16]

We start by proposing a strategy profile to show next that it is an equilibrium. The same reasons that explain Lemma 3 imply here that it is optimal for the incumbent to bid his value and for the entrants to raise their final bids to $\hat{v}(\phi(s_i))$ if the incumbent is the other top bidder. These are features of our proposed strategies. Since the event that the incumbent is not a top bidder is bad news for the entrants, our proposal is that entrants do not raise their final bids in this case. Finally, the entrants' initial bids are given by the next function.

[16]We do not make the tie-breaking rule explicit because it is irrelevant for the analysis in this section.

Definition 4. Let $\sigma : [0,1] \to [0,\infty)$ be a continuous strictly increasing function defined as follows:[17]

- If $\phi(s_i) = 0$,

$$\sigma(s_i) \equiv v(s_i, 0) \leq \hat{v}(0). \tag{7}$$

- If $\phi(s_i) > 0$ and $\int_0^1 (v(s_i, s_1) - \hat{v}(1))\, dF_1(s_1) \leq 0$, $\sigma(s_i) \in [\hat{v}(0), \hat{v}(\phi(s_i))]$ solves

$$\int_0^{\hat{v}^{-1}(\sigma(s_i))} \big(v(s_i, s_1) - \sigma(s_i)\big)\, dF_1(s_1) + \int_{\hat{v}^{-1}(\sigma(s_i))}^{\phi(s_i)} \big(v(s_i, s_1) - \hat{v}(s_1)\big)\, dF_1(s_1) = 0. \tag{8}$$

- If $\phi(s_i) > 0$ and $\int_0^1 (v(s_i, s_1) - \hat{v}(1))\, dF_1(s_1) > 0$,

$$\sigma(s_i) \equiv \int_0^1 v(s_i, s_1)\, dF_1(s_1) > \hat{v}(1). \tag{9}$$

In the first case, $\phi(s_i) = 0$ (and Definition 2) means that it is unprofitable for the entrant to outbid the incumbent. This explains that our proposed initial bid in (7) is so low that it is always outbid by the incumbent. In the last case, $\int_0^1 (v(s_i, s_1) - \hat{v}(1))\, dF_1(s_1) > 0$ means that the entrant's expected value is greater than the incumbent's maximum value. This explains that our proposed initial bid in (9) is so high that it always outbids the incumbent.

In the intermediate case, our proposed initial bid $\sigma(s_i)$ trades off the expected gains and losses of marginally outbidding the largest initial bid of the other entrants when the bidders follow the proposed strategies. The losses correspond to the first term in (8) in which the incumbent's bid $\hat{v}(s_1)$ is less than $\sigma(s_i)$. This has two implications. First, our entrant is a top bidder together with the entrant submitting the largest initial bid of the other entrants. Second, $\hat{v}(s_1) \leq \sigma(s_i)$ is bad news, which explains why entrants do not raise their final bids in our proposed strategy. Hence, our entrant wins and pays the highest initial bid of the other entrants $\sigma(s_i)$. The gains correspond to the second term in (8) in which the incumbent's bid $\hat{v}(s_1)$ is greater than $\sigma(s_i)$. This has two implications. First, our entrant and the incumbent are the top bidders. Second, $\hat{v}(s_1) \geq \sigma(s_i)$ is good news for our entrant, which explains why she raises her final bid in our proposed strategy. Since the entrant's final bid increases to $\hat{v}(\phi(s_i))$, our entrant wins at a price equal to the incumbent's final bid $\hat{v}(s_1)$ if the incumbent's type is less than $\phi(s_i)$.

Proposition 4. *The following profile of strategies is an equilibrium of our two-round auction:*

- *The incumbent bids $\hat{v}(s_1)$ in both rounds when his type is s_1.*
- *An entrant with type s_i bids $\sigma(s_i)$ in the first round.*
- *An entrant does not increase her bid in the second round if the other top bidder is another entrant.*

[17] In the Supplementary Appendix, we prove the inequalities in (7) and (9), and that the solution to (8) lies in $[\hat{v}(0), \hat{v}(\phi(s_i))]$.

- *An entrant with type s_i bids $\hat{v}(\phi(s_i))$ in the second round if the other top bidder is the incumbent.*

In the proof, we show, first, that (8) implies that an entrant has no incentive to deviate locally in the first round and, second, that no incentive to deviate locally implies no incentive to deviate globally. In the latter, we use that the entrant's payoff is supermodular in her type and initial bid when her final bid is determined optimally.

In the equilibrium of Proposition 4, entrants do not tie because their initial bids are given by a strictly increasing function and their final bids are equal to their initial bids in case the other top bidder is an entrant. Besides, the final bid of an entrant with type s_i competing in the last round with the incumbent is raised to the point of outbidding the incumbent if and only if his type is less than $\phi(s_i)$. Thus, the equilibrium implements the allocation associated to ϕ, which explains our last result.

Proposition 5. *The equilibrium in Proposition 4 implements the second best.*

7. Multiple incumbents

The simplicity of our two-round auction makes it well suited to be used in practice. Furthermore, in many applications there is only one incumbent. If there is more than one incumbent, however, only two rounds may be insufficient to aggregate the information needed to maximize efficiency and more complex mechanisms may be necessary. In this section, we explore this possibility and show how the insights of our two-round auction can be used to construct a multi-round auction that outperforms the open ascending auction. Although our multi-round auction maximizes expected social surplus in a general class, it still remains an open question whether it is second best. This requires a tractable characterization of the second best, which is still a general challenge of the auction literature.

We assume $n_I \geq 1$ incumbents and $n - n_I \geq 1$ entrants. An incumbent $j \in \{1, \dots, n_I\}$ and an entrant $i \in \{n_I + 1, \dots, n\}$ put, respectively, a monetary value of $\hat{v}(s_j)$ and of $v(s_i, s_I)$ in getting the good, where s_j, s_i, and s_I denote, respectively, the type of j, the type of i, and the vector of types of all the incumbents. We assume that s_k, $k \in \{1, \dots, n\}$, is equal to the realization of an independent random variable, with distribution F_k, density f_k, and support normalized to $[0, 1]$. We assume that both $\hat{v}$ and v are increasing functions, strictly in the case of $\hat{v}$ and the first argument of v. We also assume that v is symmetric in the last n_I arguments in the sense that $v(s_i, s_I) = v(s_i, \tilde{s}_I)$ for any permutation $\tilde{s}_I$ of s_I.

Our multi-round auction is a survival auction (see Fujishima et al. 1999) with a novel tie-breaking rule. The auction starts with all bidders active and the minimum price set to zero. In each round, the active bidders submit a sealed bid weakly larger than the current minimum price: the bidder (or bidders) submitting the lowest bid becomes inactive and her bid determines the minimum bid in the next round. Both the identity and the last bid of the bidders declared inactive are announced publicly at the end of each round. The auction ends once no more than one bidder remains active. The good is allocated to the bidder who submitted the highest bid in the last round, if there is only one. If there are several bidders, we solve the tie by selecting the bidder who placed the highest bid in the previous round. If this does not solve the tie, a uniformly random tie-breaking rule is used among the bidders who tied in the last round. In all cases, the price that the winner pays is the second highest bid in the last round.

Our survival auction combines the advantages of the open ascending auction and of our two-round auction. Indeed, the survival auction with a uniformly random tie-breaking rule is strategically equivalent to the open ascending auction. Thus, our survival auction implements the first best in the same cases as the open ascending auction because the first best does not have ties with positive probability except in nongeneric cases. However, we show next that by making use of past bids to solve ties, as in our two-round auction, the efficiency of the allocation improves if the first best is not implementable.

To do so, we study an equilibrium of the survival auction that generalizes the logic of the equilibrium of our two-round auction. We start with the following auxiliary definition that generalizes Definition 2 to our current setting.

Definition 5. For any $j \in \{1, \dots, n_I\}$, $s \in [0,1]$, and vector $s_{-j} = (s_1, \dots, s_{n_I - 1}) \in [0,1]^{n_I - 1}$ with maximum component $\underline{s}_j$, let

$$\phi_j(s; s_{-j}) \equiv \underset{s_j \geq \underline{s}_j}{\arg\max} \int_0^{s_j} \big(v(s, \tilde{s}_j, s_{-j}) - \hat{v}(\tilde{s}_j)\big)\, dF_j(\tilde{s}_j). \tag{10}$$

As in the case of ϕ, and because of the same arguments as in Lemma 2, ϕ_j is an increasing function, uniquely defined by (10) almost everywhere.

In our proposed equilibrium, incumbents use, once again, their unique weakly dominant strategy to bid their values. The entrants' strategies are, however, more sophisticated. We describe next the bid of an entrant with type s, distinguishing five cases depending on who is still active:[18]

Case A. *Three or more entrants and at least one incumbent.* The entrant bids $v(s, 0, \dots, 0)$ if all the incumbents are still active and bids $v(s, s_1, \dots, s_k, \dots, s_k)$ otherwise, where $(s_1, \dots, s_k)$ is the vector of incumbents' types inferred from the prices at which they became inactive.[19]

Case B. *Two entrants and at least two incumbents.* The entrant bids $\hat{v}(1)$.

[18] We do not describe the entrants strategy when one entrant and two or more incumbents are active because our proposed bidding in case B makes it unnecessary for our analysis: any feasible bid function works for Propositions 6 and 7.

[19] This is, $s_k = \hat{v}^{-1}(p_k)$, where p_k is the price bid by the kth incumbent withdrawing from the auction.

Case C. *Two entrants and one incumbent.* The entrant bids the $\sigma_j(s) \in [\hat{v}(\underline{s}_j), \hat{v}(\phi_j(s, s_{-j}))]$ that solves

$$\int_{\underline{s}_j}^{\hat{v}^{-1}(\sigma_j(s))} \big(v(s, s_{-j}, s_j) - \hat{v}(s_j)\big)^- \, dF_j(s_j) + \int_{\hat{v}^{-1}(\sigma_j(s))}^{\phi_j(s,s_{-j})} \big(v(s, s_{-j}, s_j) - \hat{v}(s_j)\big) \, dF_j(s_j) = 0 \tag{11}$$

for j, the identity of the incumbent still active, and $s_{-j} \equiv (s_1, \dots, s_{n_I-1})$, the vector of inferred types of the other incumbents, and $\underline{s}_j$ its maximum component.

Case D. *Only one entrant and one incumbent.* The entrant bids $\hat{v}(\phi_j(s; s_{-j}))$, where j is the identity of the incumbent still active, and s_{-j} is the vector of inferred types of the other incumbents.

Case E. *Only entrants.* The entrant bids the maximum between the minimum bid of the round and $v(s, s_1, \dots, s_{n_I})$, where $(s_1, \dots, s_{n_I})$ is the inferred vector of incumbents' types.[20]

Intuitively, entrants bid their minimum value of the good that is compatible with the information already inferred from the past play (Case A) until either only two entrants remain active (Cases B and C) or all the incumbents have quit (Case E). In Case B, entrants bid so high that no incumbent outbids them until only one incumbent remains active, Case C. In Case C, entrants' bids are similar to the bid in the first round of our two-round auction. Similarly to (8), (11) trades off the expected gains and losses of marginally outbidding the other entrant who is still active at price $\sigma_j(s)$ assuming the proposed strategies, and thus that both entrants who are still active have the same type s. The losses correspond to the first term in (11). They arise when the highest incumbent's bid $\hat{v}(s_j)$ is less than $\sigma_j(s)$. In this case, the bidding in Case E means that each entrant bids the minimum bid, which is equal to the incumbent's bid $\hat{v}(s_j)$ or her value $v(s, s_{-j}, s_j)$, whichever is larger. The tie-breaking rule means that the entrant wins in both cases, but he gets $v(s, s_{-j}, s_j) - \hat{v}(s_j) < 0$ in the former case and $v(s, s_{-j}, s_j) - v(s, s_{-j}, s_j) = 0$ in the latter. The gains are similar to those in (8). They arise when the highest incumbent's bid $\hat{v}(s_j)$ is greater than $\sigma_j(s, s_{-j})$. In this case, Case D means that the entrant competes with the incumbent in the last round and makes a positive expected profit equal to the second integral of (11).

[20] The proposed bids are always weakly greater than the minimum bid of the round. In Case E, this holds true by definition. In Case B, this is because the bid is greater than any incumbent's bid and there are incumbents still active. When the game moves from Case B to Case C, this is because the minimum bid is $\hat{v}(\underline{s}_j)$, which is less than $\sigma_j(s)$ by definition. In Cases A and D, this is because entrants submit weakly greater bids than in the previous round. This is also the case when the game moves from Case A to Case C since $\sigma_j(s) \geq v(s, s_{-j}, \underline{s}_j)$ at the solution of (11). To see why, note that the left hand side of (11) is decreasing in $\sigma_j(s)$, its first integral is zero at $\sigma_j(s) = v(s, s_{-j}, \underline{s}_j)$, and the second integral is positive by Definition 5.

Proposition 6. *There exists an equilibrium of the survival auction in which the incumbents bid their values and the entrants follow a strategy satisfying Cases A–E above.*

To understand why it is an equilibrium, note that a downward deviation in Case A means that the entrant misses the opportunity to compete at profitable prices, whereas an upward deviation allows him to compete additionally only in cases in which a higher type entrant is still active and thus willing to bid up to higher prices. The high bidding in Case B entails no risk to the entrant, as she can ensure being outbid by one incumbent by bidding the minimum bid in the next round. The reason why entrants do not have incentives to deviate in Case C is the same as in the first round of our two-round auction, where (11) substitutes (8). Finally, Definition 5 means that the bid in Case D is optimal and the fact that entrants know their values in Case E means that their bids in Case E are optimal as in a private value auction.

Proposition 7. *The equilibrium of the survival auction in Proposition 6 gives weakly greater expected social surplus than any undominated equilibrium of the open ascending auction, strictly if rushes of several bidders arise in the open ascending auction with positive probability.*

The proof of the proposition uses that our survival auction assigns the good to either the incumbent with highest type s_j or the entrant with highest type s, depending on whether s_j is greater or less than $\phi_j(s, s_{-j})$, respectively, for s_{-j}, the vector of the other incumbents' types. In the proof, we show that this allocation maximizes the expected social surplus among the allocations that satisfy two properties: that the probability with which an incumbent gets the good is increasing in his type, keeping the others types constant, and that an incumbent only gets the good if his type is the highest among the incumbents. These two properties are satisfied in the open ascending auction when incumbents use their weakly dominant strategy, which explains the proposition.

8. Conclusion

In this paper, we have shown in a setting relevant for privatizations, public tenders, and takeover contests that the information disclosed along an open ascending auction gives rise to rushes that hinder its efficiency. This result has implications that go beyond the analysis of auctions since an open ascending auction is a reasonable model of bargaining for real-life situation like takeovers. Besides, our model provides a tractable framework in which to illustrate that more transparency in the market place can be detrimental for welfare because it may lead to inefficient rushes.

One consequence of rushes that we have not explored is that they may preclude information aggregation since they occur when a set of types quits at the same price, i.e., in a semipooling equilibrium. In a more general setting than ours in which all bidders have private information about the common value component, this pooling of different types suggests the paradoxical result that increasing the transparency of the price mechanism may lead to rushes that reduce the final amount of information disclosed. This suggests that more opaque markets may aggregate more information and thus be more efficient. It may also have implications for the optimal design of an auction. We believe that the exploration of these issues is a fruitful venue for future research.

Appendix: Proofs

Proof of Lemma 1

The "only if" part follows from the fact that a necessary condition for implementability of the first best is that the probability with which the incumbent gets the good conditional on his type is an increasing function and this probability is equal to $F(\rho(s_1))^n$; see Lemma 4 in the Supplementary Appendix. The "if" part follows from the fact that ρ increasing implies that the first best is monotone and thus implementable.

Proof of Lemma 2

Since $v(s_i, s_1)$ is strictly increasing in s_1, the derivative of the maximand of (2) is single crossing in s_1, which means that there exists an increasing selection of maximizers of (2). Besides, multiple maximizers can only exist at the discontinuity points of ϕ, but since ϕ is increasing, and thus continuous a.e., ϕ is defined uniquely by (2) a.e.

Proof of Proposition 1

The proposition follows from three results: First, that the allocation associated to ϕ is implementable. Second, that the allocation associated to ϕ gives maximum expected social surplus among the allocations in Φ. Third, that there exists an allocation in Φ that gives strictly greater expected social surplus than any implementable allocation outside Φ. The first result is a direct implication of the fact that the allocation associated to ϕ is monotone. See also footnote 10. That the second result holds true is explained in the main text. To prove the third result, we show that the set of optimal solutions to a relaxed version of the problem that defines the second best lies in Φ. This relaxed problem is defined by substituting the restriction to implementable allocations with the following necessary condition for the incumbent's incentive compatibility constraint:

$$Q_1(s_1, p) \equiv \int_{[0,1]^n} p_1(s_1, s_{-1})\, dF_{-1}(s_{-1}) \quad \text{increasing in } s_1. \tag{12}$$

That this condition is necessary is shown in Lemma 4 in the Supplementary Appendix.

To show that the optimal solution for the relaxed problem belongs to Φ, we use that our relaxed problem can be solved in two stages and that any solution to the first stage lies in Φ. In this first stage, we maximize the expected social surplus, (1), with respect to p subject to the constraint $Q(\cdot, p) = \hat{Q}(\cdot)$ for an arbitrary increasing function $\hat{Q} : [0, 1] \to [0, 1]$. The second stage consists of maximizing with respect to $\hat{Q}$ the resulting outcome of the first maximization subject to $\hat{Q}$ increasing.

The first stage problem corresponding to an arbitrary increasing function $\hat{Q}$: $[0,1] \to [0,1]$ is

$$\max_{p:[0,1]^{n+1}\to\Delta(n+1)} \int_0^1 \hat{v}(s_1)\hat{Q}(s_1)\,dF_1(s_1) + \int_{[0,1]^{n+1}} \Bigl(\sum_{i\neq 1} p_i(s)v(s_i,s_1)\Bigr)\,dF_1(s_1)\,dF_{-1}(s_{-1}) \tag{13}$$
$$\text{s.t. } \int_{[0,1]^n} p_1(s_1,s_{-1})\,dF_{-1}(s_{-1}) = \hat{Q}(s_1) \quad \forall s_1\in[0,1].$$

Since $p_1(s) = 1 - \sum_{i\neq 1} p_i(s)$, the constraint of this problem can be written as

$$\int_{[0,1]^n} \sum_{i\neq 1} p_i(s_1,s_{-1})\,dF_{-1}(s_{-1}) = 1-\hat{Q}(s_1) \quad \forall s_1\in[0,1]. \tag{14}$$

We characterize the solutions to (13), leaving aside the first term of the objective function, which is constant in p, and substituting the constraint by (14). The Lagrange function is

$$\int_{[0,1]^{n+1}} \Bigl(\sum_{i\neq 1} p_i(s)v(s_i,s_1)\Bigr)\,dF_1(s_1)\,dF_{-1}(s_{-1}) - \int_0^1 \lambda(s_1)\Bigl(\int_{[0,1]^n}\sum_{i\neq 1} p_i(s_1,s_{-1})\,dF_{-1}(s_{-1}) - \bigl(1-\hat{Q}(s_1)\bigr)\Bigr)\,ds_1,$$

where $\lambda : [0,1]\to\mathbb{R}$ is the Lagrange multiplier. After some rearrangements, we get

$$\int_{[0,1]^{n+1}} \Bigl(\sum_{i\neq 1} p_i(s)\Bigl(v(s_i,s_1) - \frac{\lambda(s_1)}{f_1(s_1)}\Bigr)\Bigr)\,dF_1(s_1)\,dF_{-1}(s_{-1}) + \int_0^1 \lambda(s_1)\bigl(1-\hat{Q}(s_1)\bigr)\,ds_1.$$

The optimal Lagrange multiplier λ^* characterizes the optimal allocation of the problem in (13): the good is allocated to the entrant with the largest value $v(s_i,s_1)$ if $v(s_i,s_1) > \frac{\lambda^*(s_1)}{f_1(s_1)}$; otherwise the good is allocated to the incumbent. Thus, the optimal allocation can be characterized by a function $\zeta : [0,1]\to[0,1]$ implicitly defined by $v(\zeta(s_1),s_1) = \frac{\lambda^*(s_1)}{f_1(s_1)}$ if a solution exists and either 0 or 1, depending on whether $v(0,s_1) > \frac{\lambda^*(s_1)}{f_1(s_1)}$ or $v(1,s_1) < \frac{\lambda^*(s_1)}{f_1(s_1)}$, respectively. The good is allocated to the entrant with highest type $s_{(1)}$ if $s_{(1)} \geq \zeta(s_1)$ and otherwise to the incumbent. Consequently, to show that the optimal allocation belongs to Φ only requires one to show that ζ is an increasing function. This can be deduced from the constraint in (13). It implies that ζ satisfies $\hat{Q}_1(s_1) = F_{(1)}(\zeta(s_1)) = \prod_{i=2}^{n+1} F_i(\zeta(s_1))$, and, hence, the fact that F_i and $\hat{Q}$ are increasing implies that ζ is also increasing, as desired.

Proof of Lemma 3

Part (i) follows from the same arguments as in private value auctions: if the incumbent quits at a price lower than $\hat{v}(s_1)$, he misses the opportunity to win at profitable prices and if he remains active when the price goes above $\hat{v}(s_1)$, he risks winning when the price is higher than his value. Part (ii) can also be derived from a similar argument using the fact that when the incumbent plays his weakly dominant strategy and quits at price $\hat{v}(s_1)$, an entrant can infer that the incumbent's type is equal to s_1 and, thus, an entrant with type s_i finds it optimal to quit at price $v(s_i, s_1)$. Finally, to see why (iii) holds, note that an entrant who plans to quit at b in an information set in which the only other active bidder is the incumbent wins if and only if the incumbent's bid $\hat{v}(s_1)$ is less than b and, in this case, pays the incumbent's bid. Hence, the expected payoff of our entrant is equal to

$$\int_{s_1}^{\hat{v}^{-1}(b)} \big(v(s_i, \tilde{s}_1) - \hat{v}(\tilde{s}_1)\big) \frac{dF_1(\tilde{s}_1)}{1 - F_1(s_1)}. \tag{15}$$

By Definition 2 and Lemma 2, the unique maximizer of this function is $b = \hat{v}(\phi(s_i))$ for almost all $s_i \in [0, 1]$. To see why, note that one can get the same maximand as in (2) by multiplying (15) by $1 - F_1(s_1)$, adding $\int_0^{s_1} (v(s_i, \tilde{s}_1) - \hat{v}(\tilde{s}_1))\, dF_1(\tilde{s}_1)$ (which are constant with respect to b), and implementing the change of variable $b = \hat{v}(q)$.

Proof of Proposition 2

We argue by contradiction. We assume an equilibrium that implements the second best and show that an entrant has strict incentives to deviate.

In our argument we use two features. First, Lemma 3 pins down the incumbent's and entrants' bids a.e. for the cases to which is applicable to. The claim with respect to Lemma 3(i) is direct. The claim with respect to Lemma 3(iii) is a consequence of the uniqueness of the optimizer of (2) a.e.; see Lemma 2. The claim with respect to Lemma 3(ii) follows from the fact that there may be other optima only if a change in the bid around the proposed optimum does not change the cases in which the bidder wins. But this cannot happen in a strategy profile that implements the second best because the second best requires that the entrants use the same strictly increasing bid function after the entrant has quit.

Second, there exists a small interval of entrants' types $[s^*, s^* + \epsilon]$ for $\epsilon > 0$, an incumbent's type $s_1^* < \phi(s^*)$, and a function $\tau : [s^*, s^* + \epsilon] \to [0, s_1^*)$ such that the elements of the set

$$\big\{(s_i, s_1) : s_i \in \big[s^*, s^* + \epsilon\big], s_1 \in (\tau(s_i), s_1^*]\big\} \tag{16}$$

satisfy $v(s_i, s_1) < \hat{v}(s_1)$ and the elements of the (possibly empty) set

$$\big\{(s_i, s_1) : s_i \in \big[s^*, s^* + \epsilon\big], s_1 \in [0, \tau(s_i))\big\}$$

satisfy $v(s_i, s_1) \geq \hat{v}(s_1)$. Since ϕ is an increasing function, the second best allocates to the entrant with highest type when the vector of the highest type of the entrants and the incumbent's type lies in any of these two sets. The difference between the two sets is that the first best allocates to the incumbent in the former set and to the entrant with highest type in the latter. That this construction exists is a consequence of the conditions of the proposition.

Since the solution to (2) is unique a.e. (see Lemma 2), we can pick s^* such that $\phi(s^*)$ is the unique solution to (2), which implies that for any $\tilde{s}_1 < \phi(s^*)$,

$$\int_{\tilde{s}_1}^{\phi(s^*)} \big(v(s^*, s_1) - \hat{v}(s_1)\big)\, dF_1(s_1) > 0. \tag{17}$$

We prove the proposition by showing that an entrant with type s^* has a strictly profitable deviation: to behave as a type $s^* + \epsilon$ (according to the original strategy) until either (a) the incumbent quits, (b) all the other entrants quit, or (c) the price reaches $\hat{v}(\phi(s^*))$. When either (a) or (b) occurs first, our entrant uses the optimal strategies in Lemma 3(ii) and (iii), respectively. When (c) occurs first, the deviating entrant quits immediately and loses in the auction because she is outbid by the incumbent.

To show that the deviation is strictly profitable, we distinguish four cases. We prove that the deviation gives the same payoffs as the original strategy in the first three cases, but the deviation does strictly better than the original strategy in the last one. Thus, that there are strict incentives to deviate follows from the fact that the last case has strictly positive probability.

(i) If the highest of the other entrants' types, say $y_{(1)}$, is less than s^* and the incumbent's type s_1 is strictly less than $\phi(s^*)$, our entrant wins the auction with the original strategy and the same holds true for a type $s^* + \epsilon$. This is a consequence of Proposition 1 and the initial hypothesis that the original strategy profile implements the second best. Besides, (c) cannot occur before (a) by Lemma 3(i) since $s_1 < \phi(s^*)$. Thus, either (a) or (b) occurs at a price at which the deviating entrant is still active with the original strategy and with the deviation. Since the original strategy and the deviation prescribe the same play in both (a) and (b), the deviation gives the same payoff as the original strategy.

(ii) If $y_{(1)} < s^*$ and $\phi(s^*) < s_1$, our entrant loses the auction with the original strategy as a consequence of Proposition 1 and the initial hypothesis. She also loses after the deviation because $\phi(s^*) < s_1$ and Lemma 3(i) imply that (b) or (c) occurs before (a), and that the deviating entrant is outbid by the incumbent in either (b) or (c).

(iii) If $y_{(1)} > s^* + \epsilon$, our entrant with type s^* loses the auction with the original strategy as a consequence of Proposition 1 and the initial hypothesis. The same holds true for a type $s^* + \epsilon$. This implies that our entrant with type s^* also loses with our proposed deviation. This is direct if either (c) occurs first or the deviation means quitting before (a), (b), and (c) occur. This is also true when either (a) or (b) occurs because in either case, Lemma 3(ii) and (iii) mean that the proposed deviation prescribes quitting at a price lower than that associated to a type $s^* + \epsilon$ in the original strategy.

(iv) Suppose now that $y_{(1)} \in (s^*, s^* + \epsilon)$. The initial hypothesis and Proposition 1 means that the deviating entrant does not win the auction and, hence, gets a zero payoff when playing the original strategy. We show that the deviation gives the deviating entrant strictly positive expected profits in this case. To prove so, we distinguish three subcases:

(α) Suppose $s_1 < \tau(y_{(1)})$. We shall argue that the deviating entrant makes nonnegative payoffs. This is direct if (c) occurs first. If (a) occurs first, our entrant also loses because, first, the deviation prescribes quitting immediately after the incumbent quits at price $\hat{v}(s_1)$ if $v(s^*, s_1) < \hat{v}(s_1)$ and otherwise to quit at $v(s^*, s_1)$; second, by definition of τ, $s_1 < \tau(y_{(1)})$ means that $v(y_{(1)}, s_1) > \hat{v}(s_1)$ and $v(\cdot, s_1)$ increasing means that $v(y_{(1)}, s_1) > v(s^*, s_1)$; third, $v(y_{(1)}, s_1) > \hat{v}(s_1)$ and Lemma 3(ii) implies that the entrant with type $y_{(1)}$ quits at price $v(y_{(1)}, s_1)$. If (b) occurs first at a generic price p, that (c) has not occurred yet means that $p < \hat{v}(\phi(s^*))$, which together with Lemma 3 means that the deviating entrant remains active until price $\hat{v}(\phi(s^*))$, and wins and pays the incumbent's bid $\hat{v}(s_1)$ if $\hat{v}(s_1) < \hat{v}(\phi(s^*))$ to get

$$\int_{\hat{v}^{-1}(p)}^{\phi(s^*)} \big(v\big(s^*, s_1\big) - \hat{v}(s_1)\big)\, dF_1(s_1),$$

which is strictly positive by (17).

(β) Suppose $s_1 \in (\tau(y_{(1)}), \phi(s^*))$. We shall argue that the deviating entrant's expected payoff is at least

$$\int_{\tau(y_{(1)})}^{\phi(s^*)} \big(v\big(s^*, s_1\big) - \hat{v}(s_1)\big)\, dF_1(s_1),$$

which is strictly positive by (17), by distinguishing two possibilities:

- Suppose $s_1 \in (\tau(y_{(1)}), s_1^*]$. This implies that $v(y_{(1)}, s_1) - \hat{v}(s_1) < 0$, and, hence, that $v(s^*, s_1) - \hat{v}(s_1) < 0$ since $s^* < y_{(1)}$. This means that if (a) occurs first, the deviating bidder and all the remaining entrants quit immediately after the incumbent, the latter as a consequence of Lemma 3(ii). If the tie-breaking rule allocates the good to the deviating entrant, she gets a payoff of $v(s^*, s_1) - \hat{v}(s_1)$ as desired. Otherwise, she gets a zero payoff, which is not less than $v(s^*, s_1) - \hat{v}(s_1)$ as desired. If (b) occurs first, Lemma 3 and $s_1 < \phi(s^*)$ imply that the deviating entrant wins the auction with the deviation and gets a payoff of $v(s^*, s_1) - \hat{v}(s_1)$ as desired. Finally, (c) cannot occur first because the incumbent quits before the price reaches $\hat{v}(\phi(s^*))$ as a consequence of $s_1 < \phi(s^*)$ and Lemma 3(i).

- Suppose $s_1 \in (s_1^*, \phi(s^*))$. The same reasons as above imply that (c) cannot occur first. If (b) occurs first, the deviating entrant wins and pays $\hat{v}(s_1)$ as desired because she bids $\hat{v}(\phi(s^*))$, the incumbent bids $\hat{v}(s_1)$, and $s_1 < \phi(s^*)$. Finally, we argue that (a) cannot occur before (b) because of three reasons. First, $s_1 > s_1^*$ means that the incumbent is still active at price $\hat{v}(s_1^*)$. Second, our proposed deviation does not move out of the equilibrium path before (a), (b), or (c) is reached. Third, any equilibrium that implements the second best satisfies along the equilibrium path that only one entrant remains active at price $v(s_1^*)$ if the entrants' types are less than $s^* + \epsilon$ and the incumbent is still active. This latter property follows from the fact that the second best is incompatible with a tie and that Lemma 3(ii) means that the remaining entrants quit immediately if the incumbent quits at price $\hat{v}(s_1^*)$ since $v(x, s_1^*) - \hat{v}(s_1^*) < 0$ for any $x < s^* + \epsilon$ because $(s^* + \epsilon, s_1^*)$ belongs to the set in (16).

(γ) Suppose $s_1 > \phi(s^*)$. We shall argue that the deviating entrant makes a zero payoff. This is straightforward if (c) occurs first. If (b) occurs first, Lemma 3(i) and $s_1 > \phi(s^*)$ mean that the incumbent outbids the deviating bidder and, hence, the latter gets a zero payoff. Finally, (a) cannot occur first because (c) occurs before (a) since $s_1 > \phi(s^*)$. □

Proof of Proposition 3

Since our proposed strategies are derived from Lemma 3, we only need to check for an entrant's unilateral deviation to a bid[21] $b \in [\hat{v}(0), \hat{v}(1)]$ in information sets in which no bidder has quit yet, assuming that the deviating entrant plays optimally in the continuation game. This deviation gives zero payoffs except in the following cases.

If the other entrant quits first, Lemma 3 and $\phi(s_i) = 1$ mean that the deviating entrant wins at a price fixed by the incumbent's bid, and thus gets[22]

$$\int_0^{\gamma^{-1}(\hat{v}^{-1}(b))} \int_{\gamma(s_j)}^1 \big(v(s_i, s_1) - \hat{v}(s_1)\big)\, dF_1(s_1)\, dF(s_j). \tag{18}$$

If the incumbent quits first and the deviating entrant quits immediately after because $v(s_i, s_1) - \hat{v}(s_1) \le 0$ (see Lemma 3(ii)), our entrant wins with probability $1/2$ after a tie if the other entrant also quits after the incumbent, which occurs when her type is less than $\rho(s_1)$. Thus, our entrant gets[23]

$$\int_0^{\hat{v}^{-1}(b)} \int_{\min\{\gamma^{-1}(s_1), \rho(s_1)\}}^{\rho(s_1)} \frac{1}{2}\big(v(s_i, s_1) - \hat{v}(s_1)\big)^-\, dF(s_j)\, dF_1(s_1). \tag{19}$$

If the incumbent quits first and the deviating entrant does not quit immediately after because $v(s_i, s_1) - \hat{v}(s_1) > 0$, the deviating entrant wins and pays the other entrant's bid when the deviating entrant's value $v(s_i, s_1)$ is greater than the other entrant's bid $\max\{\hat{v}(s_1), v(s_j, s_1)\}$. Our entrant gets

$$\int_0^{\hat{v}^{-1}(b)} \int_{\gamma^{-1}(s_1)}^1 \big(v(s_i, s_1) - \max\{\hat{v}(s_1), v(s_j, s_1)\}\big)^+\, dF(s_j)\, dF_1(s_1). \tag{20}$$

[21] The restriction to $[\hat{v}(0), \hat{v}(1)]$ is without loss of generality since $b < \hat{v}(0)$ and $b > \hat{v}(1)$ give, respectively, the same expected payoff as $\hat{v}(0)$ and $\hat{v}(1)$.

[22] We adopt the convention that $\gamma^{-1}(s_1) = \bar{s}$ for $s_1 \in [\gamma(\bar{s}), 1]$.

[23] We denote $(a)^- \equiv \min\{a, 0\}$ and $(a)^+ \equiv \max\{a, 0\}$ for any $a \in \mathbb{R}$.

The sum of (18)–(20) is (i) supermodular in (s_i, b), (ii) has a derivative with respect to b equal to zero for $s_i \in [0, \bar{s}]$ and $b = \hat{v}(\gamma(s_i))$, and (iii) its derivative with respect to b is equal to zero for $b \in [\hat{v}(\gamma(\bar{s})), \hat{v}(1)]$ and $s_i = \bar{s}$. Statement (i) is straightforward; (ii) follows from (5) and $v(s_i, \gamma(s_i)) - \hat{v}(\gamma(s_i)) < 0$ for $s_i \in [0, \bar{s}]$; (iii) follows because each of the equations (18)–(20) is constant in b for $b \in [\hat{v}(\gamma(\bar{s})), \hat{v}(1)]$ and $s_i = \bar{s}$—(18) because $\gamma^{-1}(b) = \bar{s}$, (19) because $v(\bar{s}, s_1) - \hat{v}(s_1) \geq 0$ for any $s_1 \geq \gamma(\bar{s})$ since $v(\bar{s}, \gamma(\bar{s})) - \hat{v}(\gamma(\bar{s})) = 0$ and we assume (IV), and (20) because $\gamma^{-1}(s_1) = \bar{s}$ for $s_1 \in [\gamma(\bar{s}), 1]$.

Statements (i) and (ii) imply that the sum of (18)–(20) evaluated at $s_i \in [0, \bar{s}]$ is nonincreasing in b for $b \in [\hat{v}(\gamma(s_i)), \hat{v}(\gamma(\bar{s}))]$. Furthermore, (i) and (iii) imply that the sum of (18)–(20) evaluated at $s_i \in [0, \bar{s}]$ is nonincreasing in b for $b \in [\hat{v}(\gamma(\bar{s})), \hat{v}(1)]$. Thus, one can conclude that for any $s_i \in [0, \bar{s}]$, the sum of (18)–(20) is nonincreasing in b for $b \in (\hat{v}(\gamma(s_i), \hat{v}(1)]$. By a symmetric argument, one can also conclude that for any $s_i \in [0, 1]$, the sum of (18)–(20) is nondecreasing in b for $b \in [\hat{v}(0), \hat{v}(\gamma(s_i)))$. This implies that for any $s_i \in [0, 1]$, $b = \hat{v}(\gamma(s_i))$ maximizes the sum of (18)–(20) as desired.

Proof of Proposition 4

As in a private value second price auction, the incumbent finds it weakly dominant to bid his value in both rounds, and hence has no incentive to deviate. A similar argument as in Lemma 3 means here that an entrant who follows our proposed strategy in the first round does not have incentives to deviate unilaterally in the second round when the other top bidder is the incumbent.

Next, we show that an entrant with type s_i who follows our proposed strategy in the first round does not have incentives to deviate unilaterally in the second round when the other top bidder is another entrant. Since we propose that the final bid in this case is equal to the initial bid, a deviation consists of raising the final bid. This is not profitable if our entrant's expected value conditional on the incumbent not being a top bidder is less than our entrant's initial bid, i.e.,

$$\int_0^{\hat{v}^{-1}(\sigma(s_i))} \big(v(s_i, s_1) - \sigma(s_i)\big)\, dF_1(s_1) \leq 0. \tag{21}$$

We show that this inequality is satisfied by the definition of σ. In the third case of Definition 4, (21) follows from (9) since $\sigma(s_i) > \hat{v}(1)$ means that $\hat{v}^{-1}(\sigma(s_i)) = 1$. In the second case of Definition 4, (21) follows because its left hand side is equal to the first integral of (8), the second integral of (8) is positive by (2), and the left hand side of (8) is equal to zero. This completes our argument because in the first case of Definition 4, (7) means that the incumbent bids higher than our entrant and thus the incumbent must necessarily be the other top bidder.

Finally, we show that an entrant with type s_i does not find it optimal to deviate unilaterally from his first round bid $\sigma(s_i)$ to a bid b_0, assuming that she bids optimally in the second round. We compute the payoffs of this deviation by distinguishing three cases.

(i) Our entrant submits the highest initial bid of the entrants and the other top bidder is another entrant. In this case, our entrant wins and pays the initial bid of the other top bidder because the other bidder is an entrant who does not raise her final bid. Thus, our entrant's payoff is

$$\int_0^{\sigma^{-1}(b_0)} \int_0^{\hat{v}^{-1}(\sigma(y_{(1)}))} \big(v(s_i, s_1) - \sigma(y_{(1)})\big)\, dF_1(s_1)\, dG_{(1)}(y_{(1)}), \tag{22}$$

where $G_{(1)}(y_{(1)})$ denotes the distribution of the highest type of the other entrants, and where we adopt the convention that $\hat{v}^{-1}(x) = 1$ for $x > \hat{v}(1)$ and $\hat{v}^{-1}(x) = 0$ for $x < \hat{v}(0)$.

(ii) Our entrant submits the highest initial bid of the entrants and the other top bidder is the incumbent. In this case, our entrant wins if her final bid b_I outbids the incumbent's and pays the incumbent's final bid. Thus, our entrant's payoff is

$$\max_{b_I \geq b_0} \int_0^{\sigma^{-1}(b_0)} \int_{\hat{v}^{-1}(\sigma(y_{(1)}))}^{\hat{v}^{-1}(b_I)} \big(v(s_i, s_1) - \hat{v}(s_1)\big)\, dF_1(s_1)\, dG_{(1)}(y_{(1)}), \tag{23}$$

where the maximizer of this expression as a function of b_0 is denoted by $b_I^*(b_0)$.

(iii) Our entrant is a top bidder because she outbids the incumbent and submits the second highest initial bid of the entrants. In this case, the entrant wins if her final bid b_E outbids the initial bid of the other top bidder because the other bidder is an entrant who does not raise her final bid. Thus, our entrant's payoff is

$$\begin{aligned} &\max_{b_E \geq b_0} \int_{\sigma^{-1}(b_0)}^{\sigma^{-1}(b_E)} F_{(2)}\big(\sigma^{-1}(b_0) | y_{(1)}\big) \\ &\quad \times \left(\int_0^{\hat{v}^{-1}(b_0)} \big(v(s_i, s_1) - \sigma(y_{(1)})\big)\, dF_1(s_1) \right) dG_{(1)}(y_{(1)}), \end{aligned} \tag{24}$$

where $F_{(2)}(x | y_{(1)})$ denotes the probability that the second highest type of the other entrants is less than x conditional on the highest type of the other entrants equal to $y_{(1)}$. The first order conditions of (24) imply that the maximizer of (24) as a function of b_0 is

$$b_E^*(b_0) \equiv \max\left\{ b_0, \frac{\int_0^{\hat{v}^{-1}(b_0)} v(s_i, s_1)\, dF_1(s_1)}{F_1\big(\hat{v}^{-1}(b_0)\big)} \right\}. \tag{25}$$

To finish the proof we show that the sum of (22), (23), and (24) is maximized at $b_0 = \sigma(s_i)$. To do so, we check the following necessary and sufficient conditions of optimality: (i) the derivative of the sum of (22), (23), and (24) with respect to b_0 at $b_0 = \sigma(s_i)$ is equal to zero, and (ii) the derivative of the sum of (22), (23), and (24) with respect to s_i increases in b_0, i.e., the resulting function is supermodular in (s_i, b_0).

To prove (i), note the following facts:

- The derivative of (22) with respect to b_0 at $b_0 = \sigma(s_i)$ is

$$\left(\int_0^{\hat{v}^{-1}(\sigma(s_i))} \big(v(s_i, s_1) - \sigma(s_i)\big)\, dF_1(s_1)\right) \cdot G'_{(1)}(s_i) \cdot \big(\sigma^{-1}\big)'\big(\sigma(s_i)\big). \tag{26}$$

- The derivative of (23) is, by the envelope theorem,

$$\left(\int_{\hat{v}^{-1}(\sigma(s_i))}^{\phi(s_i)} \big(v(s_i, s_1) - \hat{v}(s_1)\big)\, dF_1(s_1)\right) \cdot G'_{(1)}(s_i) \cdot \big(\sigma^{-1}\big)'\big(\sigma(s_i)\big), \tag{27}$$

  since $b_I^*(\sigma(s_i)) = \hat{v}(\phi(s_i))$ because both maximizations in (2) and (23) have the same solution when $\sigma(s_i) \le \hat{v}(\phi(s))$.

- The derivative of (24) with respect to s_i at $b_0 = \sigma(s_i)$ is zero since (21) means that $b_E^*(\sigma(s_i)) = \sigma(s_i)$ by (25).

Thus, (i) follows from applying to the sum of (26) and (27) the definition of $\sigma(s_i)$ in Definition 4.

To prove (ii), we use that the derivative of the sum of (22), (23), and (24) with respect to s_i is, by the envelope theorem,

$$\begin{aligned}&\int_0^{\sigma^{-1}(b_0)} \int_0^{b_I^*(b_0)} \frac{\partial v(s_i, s_1)}{\partial s_i}\, dF_1(s_1)\, dG_{(1)}(y_{(1)}) \\ &\quad + \int_{\sigma^{-1}(b_0)}^{\sigma^{-1}(b_E^*(b_0))} F_{(2)}\big(\sigma^{-1}(b_0)|y_{(1)}\big)\left(\int_0^{\hat{v}^{-1}(b_0)} \frac{\partial v(s_i, s_1)}{\partial s_i}\, dF_1(s_1)\right) dG_{(1)}(y_{(1)}).\end{aligned}$$

To prove that this expression is increasing in b_0, note that $\hat{v}$, v, and σ are increasing functions. Thus, it is sufficient to show that b_I^* and b_E^* are also increasing. The former is direct from its definition in (23) and the latter from (25).

Proof of Proposition 5

Our equilibrium has the following features: (a) entrants initial bids are determined by the same strictly increasing function σ; (b) an entrant's final bid is equal to her initial bid if the other top bidder is an entrant; (c) an entrant's final bid is equal to $\hat{v}(\phi(s_i))$ if the other top bidder is the incumbent; (d) $\phi(s_i) < 1$ implies that $\sigma(s_i) \le \hat{v}(\phi(s_i))$;[24] (e) the incumbent uses the same strictly increasing function $\hat{v}$ in both rounds. Feature (a) implies that the entrant with the largest type $s_{(1)}$ is always a top bidder. This entrant wins the auction if $\phi(s_{(1)})$ is greater than the incumbent's type as a consequence of (b), (c), and (e). Features (a), (d), and (e) imply that if $\phi(s_{(1)})$ is less than the incumbent's type, then the incumbent is the other top bidder. Besides, if $\phi(s_{(1)})$ is less than

[24] Note by Definition 2 that $\phi(s_1) < 1$ implies that $\int_{\phi(s_i)}^1 (v(s_i, s_1) - \hat{v}(s_1))\, dF_1(s_1) \le 0$, which implies that $\int_{\phi(s_i)}^1 (v(s_i, s_1) - \hat{v}(1))\, dF_1(s_1) < 0$, since $\hat{v}$ is increasing. Thus $\int_0^1 (v(s_i, s_1) - \hat{v}(1))\, dF_1(s_1) < 0$ since $v(s_i, s_1)$ is increasing in s_1. Consequently, $\phi(s_i) < 1$ means that σ is defined by the first two cases in Definition 4 in which $\sigma(s_i) \le \hat{v}(\phi(s_i))$.

the incumbent's type, (c) and (e) imply that the incumbent wins. Thus, our equilibrium implements the allocation associated to ϕ and, thus, it is second best by Proposition 1.

Proof of Proposition 6

Since incumbents play their weakly dominant strategy, they do not have incentives to deviate. Next, we study the incentives of an entrant to deviate unilaterally from the proposed strategies. An adaptation of the argument that bidding one's value, if known, is optimal means that the entrant does not have an incentive to deviate in Case E. Definition 5 means that the bid in Case D is optimal. Next, we show that there are no incentives to deviate in Case C with a similar argument as in Proposition 4.

We start by computing the payoffs of an entrant with type s after a unilateral deviation to bid $\hat{b}$ assuming that she bids optimally in the last round. We denote by s_j the type of the incumbent still active and by s_k the type of the other entrant active. To simplify the notation we drop the dependence on the vector of types of the other incumbents inferred from their bids. We let $\underline{s}_j$ and $\underline{s}_k$ be the highest type of the incumbents and of the entrants, respectively, who have already quit. We distinguish three cases.

(a) Case $\hat{b} > \sigma_j(s_k) > \hat{v}(s_j)$. This is, our entrant bids higher than the other entrant and the other entrant bids higher than the incumbent. In this case, both entrants compete in the last round and the minimum bid is set by the incumbent's bid $\hat{v}(s_j)$. This gives expected payoffs[25]

$$\begin{aligned} &\int_{\underline{s}_j}^{\hat{v}^{-1}(\hat{b})} \max_{b \geq \hat{v}(s_j)} \int_{\sigma_j^{-1}(\hat{v}(s_j))}^{\sigma_j^{-1}(\hat{b})} 1_{b \geq \max\{v(s_k, s_j), \hat{v}(s_j)\}} \\ &\quad \times \bigl(v(s, s_j) - \max\bigl\{v(s_k, s_j), \hat{v}(s_j)\bigr\}\bigr)\, dF_k(s_k)\, dF_j(s_j), \end{aligned} \tag{28}$$

since our deviating entrant wins in case of a tie and the other entrant bids $\max\{v(s_k, s_j), \hat{v}(s_j)\}$ in the last round.

(b) Case $\sigma_j(s_k) > \hat{b} > \hat{v}(s_j)$. This is, our entrant bids less than the other entrant but more than the incumbent. As above, both entrants compete in the last round and the minimum bid is set by the incumbent's bid $\hat{v}(s_j)$. This gives expected payoffs similar to those above,

$$\begin{aligned} &\int_{\underline{s}_j}^{\hat{v}^{-1}(\hat{b})} \max_{b \geq \hat{v}(s_j)} \int_{\sigma_j^{-1}(\hat{b})}^{1} 1_{b > \max\{v(s_k, s_j), \hat{v}(s_j)\}} \\ &\quad \times \bigl(v(s, s_j) - \max\bigl\{v(s_k, s_j), \hat{v}(s_j)\bigr\}\bigr)\, dF_k(s_k)\, dF_j(s_j); \end{aligned} \tag{29}$$

the only difference is that in case of a tie, it is the other entrant who wins.

(c) Case $\hat{b} > \sigma_j(s_k)$ and $\hat{v}(s_j) > \sigma_j(s_k)$. This is, the other entrant submits the lowest bid. In this case, the deviating entrant and the incumbent compete in the last

[25] The term 1_X is an indicator function that takes value 1 if X is satisfied and 0 otherwise.

round and the minimum bid is set by the other entrant's bid $\sigma_j(s_k)$. This gives expected payoffs

$$\int_{\underline{s}_k}^{\sigma_j^{-1}(\hat{b})} \max_{b \geq \sigma_j(s_k)} \int_{\hat{v}^{-1}(\sigma_j(s_k))}^{\hat{v}^{-1}(b)} \big(v(s, s_j) - \hat{v}(s_j)\big)\, dF_j(s_j)\, dF_k(s_k), \tag{30}$$

since the incumbent bids in the last round $\hat{v}(s_j)$, and thus ties do not occur with positive probability and the deviating entrant wins if and only if $b \geq \hat{v}(s_j)$, i.e., $s_j \leq \hat{v}^{-1}(b)$.

Next, we show that the sum of (28), (29), and (30) is maximized at $\hat{b} = \sigma_j(s)$ for any $s \geq \underline{s}_k$. To do so, we check the following necessary and sufficient conditions of optimality: (i) the derivative of the sum of (28), (29), and (30) with respect to $\hat{b}$ evaluated at $\hat{b} = \sigma_j(s)$ is equal to zero; (ii) the derivative of the sum of (28), (29), and (30) with respect to s increases in $\hat{b}$, i.e., the sum of the sum of (28), (29), and (30) is supermodular in $(s, \hat{b})$.

To prove (i), we use the envelope theorem to show the following results:

- The derivative of (28) with respect to $\hat{b}$ at $\hat{b} = \sigma_j(s)$ is equal to $\frac{F_k'(s)}{\sigma_j'(s)}$ times

$$\int_{\underline{s}_j}^{\hat{v}^{-1}(\sigma_j(s))} 1_{b^*(s_j) \geq \max\{v(s,s_j), \hat{v}(s_j)\}} \big(v(s, s_j) - \max\{v(s, s_j), \hat{v}(s_j)\}\big)\, dF_j(s_j), \tag{31}$$

  where $b^*(s_j) = \max\{v(s, s_j), \hat{v}(s_j)\}$ is the optimizer of the maximization in (28). Thus, (31) is equal to

$$\int_{\underline{s}_j}^{\hat{v}^{-1}(\sigma_j(s))} \big(v(s, s_j) - \hat{v}(s_j)\big)^{-}\, dF_j(s_j). \tag{32}$$

- The derivative of (29) with respect to $\hat{b}$ at $\hat{b} = \sigma_j(s)$ is equal to $\frac{F_j'(\hat{v}^{-1}(\sigma_j(s)))}{\hat{v}'(\hat{v}^{-1}(\sigma_j(s)))}$ times

$$\int_{s_j}^{1} 1_{b^*(s_j) > \max\{v(s_k,s_j), \sigma_j(s)\}} \big(v(s, s_j) - \max\{v(s_k, s_j), \sigma_j(s)\}\big)\, dF_k(s_k) \tag{33}$$

  (for $s_j = \hat{v}^{-1}(\sigma_j(s))$) plus $\frac{F_k'(s)}{\sigma_j'(s)}$ times

$$-\int_{\underline{s}_j}^{\hat{v}^{-1}(\sigma_j(s))} 1_{b^*(s_j) > \max\{v(s,s_j), \hat{v}(s_j)\}} \big(v(s, s_j) - \max\{v(s, s_j), \hat{v}(s_j)\}\big)\, dF_j(s_j), \tag{34}$$

  where $b^*(s_j) = \max\{v(s, s_j), \hat{v}(s_j)\}$ is the optimizer of the maximization in (29). Thus, (33) and (34) are equal to zero.

- The derivative of (30) with respect to $\hat{b}$ at $\hat{b} = \sigma_j(s)$ is equal to $\frac{F_k'(s)}{\sigma_j'(s)}$ times

$$\int_{\hat{v}^{-1}(\sigma_j(s))}^{\hat{v}^{-1}(b^{**})} \big(v(s, s_j) - \hat{v}(s_j)\big)\, dF_j(s_j), \tag{35}$$

where $b^{**} = \hat{v}(\phi_j(s))$ is the optimizer of the maximization in (30). Thus, (35) is equal to

$$\int_{\hat{v}^{-1}(\sigma_j(s))}^{\phi_j(s)} \big(v(s, s_j) - \hat{v}(s_j)\big)\, dF_j(s_j). \tag{36}$$

Hence, (i) follows from applying the definition of σ_j in (11) to the sum of (32) and (36). Recall that, for notational convenience, we left aside the dependence on s_{-j}.

To prove (ii), note that the derivative of (30) with respect to s is increasing in $\hat{b}$ by the envelope theorem. Next, note that the envelope theorem also implies that the derivative of the sum of (28) and (29) with respect to s is equal to the sum of the increasing function of $\hat{b}$,

$$\begin{aligned} &\int_{\underline{s}_j}^{\hat{v}^{-1}(\hat{b})} \int_{\sigma_j^{-1}(\hat{v}(s_j))}^{1} 1_{b^*(s_j) > \max\{v(s_k, s_j), \hat{v}(s_j)\}} \frac{\partial v(s, s_j)}{\partial s}\, dF_k(s_k)\, dF_j(s_j) \\ &\quad + \int_{\underline{s}_j}^{\hat{v}^{-1}(\hat{b})} \int_{\sigma_j^{-1}(\hat{v}(s_j))}^{\hat{v}^{-1}(\hat{b})} 1_{b^*(s_j) = \max\{v(s_k, s_j), \hat{v}(s_j)\}} \frac{\partial v(s, s_j)}{\partial s}\, dF_k(s_k)\, dF_j(s_j), \end{aligned}$$

where $b^*(s_j) = \max\{v(s, s_j), \hat{v}(s_j)\}$ is the optimizer of the maximizations in (28) and (29).

Consequently, we can conclude that there are no incentives to deviate in Case C.

To show that there are no strict incentives to deviate in Case A or in Case B, it is sufficient to show that the continuation payoffs are (α) nonnegative and (β) equal to zero if the deviating entrant's type is less than the second highest type of the other entrants $\underline{s}_k$. This is because a downward deviation in either Case A or Case B only means that the deviating entrant may miss the opportunity to bid in Case C or in Case E, and an upward deviation in Case A (in Case B it is not payoff relevant) only means that the entrant can bid in Case C or Case E in some additional cases in which the second highest type of the other entrants is larger than the type of the deviating entrant.

Property (α) is satisfied when we move from Case A to Case E because the deviating bidder can guarantee zero payoffs by bidding the minimum bid in the round. This bid loses with probability 1 because the minimum bid is fixed by the minimum bid in the round in Case A and the other entrants who are still active weakly increase their bids from the round in Case A to the round in Case E. Property (β) is satisfied when we move from Case A to Case E because the deviating entrant cannot find it profitable to outbid in Case E an entrant with a larger type that follows our proposed bids.

That (α) is satisfied when we move from Case A or B to Case C (or from Case A to Case B and then to Case C) is similar. The deviating bidder can guarantee zero payoffs by bidding the minimum bid in the round, in this case, because the incumbent who is still active bids with probability 1 above the minimum bid. To show that (β) is satisfied when we move from Case A or B to Case C (or from Case A to Case B and then to Case C), we use that the continuation payoffs of a deviating entrant with type s are equal to the sum of (29), (30), and (28) evaluated at the value of $\hat{b}$ that maximizes this sum. Hence, (β) can be deduced from the following two features. First, the continuation payoffs of an entrant with type s is zero if $s = \underline{s}_k$. To see why, note that we have already argued that the value of $\hat{b}$ that maximizes the sum of (29), (30), and (28) for $s = \underline{s}_k$ is $\hat{b} = \sigma_j(\underline{s}_k)$. In this case, (29) and (30) are equal to zero, and (28) does not apply because $\hat{b} > \sigma_j(s_k)$ does not hold for any $s_k > \underline{s}_k$ and $\hat{b} = \sigma_j(\underline{s}_k)$. Second, the continuation payoffs are weakly increasing in s. This can be shown by applying the envelope theorem to the sum of (29), (30), and (28) evaluated at the optimal $\hat{b}$.

Proof of Proposition 7

We first note that the unique weakly dominant strategy of the incumbents in the open ascending auction is to bid their values, and that all incumbents have the same strictly increasing value function. This implies that any equilibrium in weakly undominated strategies of the open ascending auction satisfies two properties: an incumbent is assigned the good only if he has the largest type of the incumbents, and an incumbent who gets the good assigned for a certain type also gets the good assigned for larger types if the other bidders' types remain constant.

Next, note that our survival auction implements the allocation associated to $\{\phi_j\}_{j\in I}$; see Lemma 5 in the Supplementary Appendix. This is the good assigned to either the incumbent with he highest type s_j or the entrant with the highest type s depending on whether s_j is greater or less than $\phi_j(s, s_{-j})$, respectively, for s_{-j}, the vector of the other incumbents' types.

Consequently, to prove the proposition, we show (i) that the allocation associated to $\{\phi_j\}_{j\in I}$ maximizes the expected social surplus among the allocations that satisfy the two properties above, and (ii) that the optimal allocations in this restricted set never assign with positive probability to entrants who do not have the largest type. Part (ii) is direct: for any allocation in the restricted set that allocates to an entrant whose type is not the highest, the allocation that instead allocates to the entrant with the highest type gives strictly higher expected social surplus and belongs to the restricted set. To prove (i), we use an adaptation of the argument in the paragraph before Proposition 1.

Any allocation that satisfies the above two properties and that only allocates to an entrant if she has the largest type can be characterized with cutoffs for the highest type of the incumbent as a function of the other bidders' types such that the good is allocated to the incumbent only if it is above the cutoff and otherwise to the entrant with the highest type. There is no loss in assuming that the cutoffs do not depend on the entrants' types but the highest types. This is so as the other entrants types do not affect the comparison between allocating to the incumbent with the highest type or to the entrant with the highest type. We denote the cutoffs with a set of functions $\{\hat{\phi}_j\}_{i\in I}$, where $\hat{\phi}_j(s; s_{-j})$ is the cutoff for incumbent j when s_{-j} is the vector of other incumbents' types and s is the the highest of the entrants'types. The expected social surplus of the allocation with cutoffs given by $\{\hat{\phi}_j\}_{i\in I}$ is equal to

$$\sum_{j\in I} \int_{[0,1]^{n_I-1}} \int_0^1 \left(\int_{\underline{s}_j(s_{-j})}^{\hat{\phi}_j(s;s_{-j})} v(s; s_{-j}, s_j)\, dF_j(s_j) + \int_{\hat{\phi}_j(s;s_{-j})}^1 \hat{v}(s_j)\, dF_j(s_j) \right) dF_{(1)}(s)\, dF_{-j}(s_{-j}), \tag{37}$$

where $\underline{s}_j(s_{-j})$ is equal to the largest element of s_{-j}, $F_{(1)}(s)$ is the distribution of the largest type of the entrants, and $F_{-j}(s_{-j})$ is the distribution of the vector of the incumbents' types s_{-j} where incumbent j has been excluded.

Since (37) is equal to

$$\sum_{j\in I}\int_{[0,1]^{n_I-1}}\int_0^1\int_{\underline{s}_j(s_{-j})}^{\hat{\phi}_j(s;s_{-j})}\big(v(s;s_{-j},s_j)-\hat{v}(s_j)\big)\,dF_j(s_j)\,dF_{(1)}(s)\,dF_{-j}(s_{-j})$$
$$+\sum_{j\in I}\int_{[0,1]^{n_I-1}}\int_0^1\int_{\underline{s}_j(s_{-j})}^{1}\hat{v}(s_j)\,dF_j(s_j)\,dF_{(1)}(s)\,dF_{-j}(s_{-j}),$$

Definition 5 means that $\{\hat{\phi}_j(s;s_{-j}))\}_j=\{\phi_j(s;s_{-j}))\}_j$ maximizes (37), which implies the proposition.

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